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Single-sided domain walls in M-theory

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Abstract

We describe some single-sided BPS domain wall configurations in M-theory. These are smooth non-singular resolutions of Calabi–Yau orbifolds obtained by identifying the two sides of the wall under reflection. They may thus be thought of as domain walls at the end of the universe. We also describe related domain wall type solutions with a negative cosmological constant. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

It is now widely recognized that topological defects with p spatial dimensions invariant under half the maximum number of supersymmetries — BPS configurations — play a central role in non-perturbative string theory and M-theory. If p is less than $(n - 3)$, where n is the space–time dimension, then such objects can be studied in at least two limits. One is the *light* approximation in which the gravitational field that the objects generate is ignored. Treated classically, the world-volume theory of such objects is described by a Dirac–Born–Infeld type action. The other is the *heavy* approximation, in which the gravitational field generated by the objects is taken into account and one looks for solutions of the supergravity equations of motion.

If $p < n - 3$ heavy branes give rise to asymptotically flat metrics in directions transverse to the brane, and from a distance they behave more or less like light branes moving in a flat

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background. If $p = n - 3$ (vortices) or $p = n - 2$ (domain walls), however, the metrics they generate are not asymptotically flat. For vortices — for example, the 7-brane of the 10-dimensional type IIB theory — the metric has an angular deficit. In the case of domain walls, their effect on space–time can be even more drastic. For example, conventional domain walls, even in the thin-wall approximation, bring about the compactification of space [1]. This happens as follows. Either side of the domain wall is isometric to the interior of a time-like hyperboloid in Minkowski space–time $\mathbb{E}^{n-1,1}$. To get the entire space–time one glues two such domain walls back to back. The induced metric is continuous across the domain wall but the second fundamental form has a discontinuity which gives the distributional stress tensor. Another feature of conventional domain walls, which is more or less obvious from the description just given, is that one does not expect to have more than one in a static configuration.

Domain walls of an unconventional (orbifold) type play an important role in the approach of Hořava and Witten [2,3] approach to the $E_8 \times E_8$ heterotic string theory in \mathbf{M} -theory. They also have a drastic global effect on the structure of space–time. Of course in addition to their gravitational fields one must take into account the effects of anomalies and the four-form field strength.

In this paper we are going to study the global structure of some other space–times containing BPS domain walls that have arisen in \mathbf{M} -theory. A striking feature of \mathbf{M} -theory is the extent to which configurations in (11)-dimensions are non-singular even though they may appear to be singular in lower dimensions. We shall therefore be particularly interested everywhere in non-singular configurations.

2. Bianchi domain walls

We shall consider p -brane solutions of the form

$$M_4 \times \mathbb{E}^{p-3,1},$$

where M_4 is a non-compact Riemannian four-manifold which is either Ricci-flat or has negative cosmological constant. If $p = 3$ we would be considering domain walls in five space–time dimensions. We are looking for metrics on M_4 which depend only on one coordinate t , transverse to the domain wall. The metric should be homogeneous in the directions parallel to the wall. Mathematically this means that we are looking for cohomogeneity 1, or hypersurface homogeneous, metrics invariant under the action of Lie group G acting transitively on three-dimensional orbits. In the cases we are interested in G may be taken to be three-dimensional and the possible groups have been classified by Bianchi (see e.g. [4]). The problem is very similar to that encountered in studying homogeneous Lorentzian cosmologies and we shall freely use standard results from that subject [5]. The Bianchi types relevant to this paper are type I, II, VI₀ and VII₀. Domain walls of types I and II are discussed in Section 3, while the treatment of the more “exotic” solutions is relegated to Section 6.

Table 1
Some of the Bianchi types

Bianchi type	n_1	n_2	n_3	Group of motions
I	0	0	0	\mathbb{R}^3
II	0	0	1	Nil
VI ₀	1	-1	0	$E(1, 1)$
VII ₀	1	1	0	$E(2)$

In the following we shall find that all Ricci-flat solutions are singular and describe how the singularity of the type II solution may be resolved. The resolution of this singularity gives a complete Ricci-flat Kählermanifold that we shall call the BKTY metric.¹

In the light of the recently proposed AdS/CFT correspondence [9–11] it is instructive to investigate domain walls in the Anti-de Sitter background. Sections 5 and 6.3 are therefore devoted to the study of four-manifolds of the above-mentioned Bianchi types with negative cosmological constant.

2.1. Bianchi models

Let us now be more specific about the four-manifolds M_4 in question. Spaces of interest are homogeneous manifolds with the following ansatz for the metric:

$$ds^2 = dt^2 + a^2(t)(\sigma^1)^2 + b^2(t)(\sigma^2)^2 + c^2(t)(\sigma^3)^2. \tag{1}$$

Here t is the imaginary time and the metric coefficients are functions of t only. The one-forms $\{\sigma^k\}$, $k = 1, 2, 3$, are left-invariant one-forms of the three-dimensional group of isometric motions G and as such satisfy

$$d\sigma^k = -\frac{1}{2}n_k \epsilon_{ijk} \sigma^i \wedge \sigma^j, \quad \text{no sum over } k,$$

where constants $\{n_k\}$ are the structure constants of G . The four-manifolds may be classified according to the group of isometric motions. This is the Bianchi classification in which each type corresponds to a particular set of values of the structure constants $\{n_k\}$. In the rest of the paper manifolds of four Bianchi types will arise, whose properties are summarized in Table 1. Note that all four groups of isometric motions are solvable, in fact they all have one non-trivial commutator. The Einstein equations for the metric (1) reduce to the following set of second-order ODEs:

$$- - R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c}, \tag{2}$$

$$R_1^1 = \frac{(\dot{a}bc)'}{abc} + \frac{1}{2} \frac{1}{a^2 b^2 c^2} [n_1^2 a^4 - (n_2 b^2 - n_3 c^2)^2], \tag{3}$$

¹ The name BKTY is derived from the initials of the authors of [6–8] who constructed this space as a certain degeneration of the K3 surface.

$$R_2^2 = \frac{(\dot{abc})}{abc} + \frac{1}{2} \frac{1}{a^2 b^2 c^2} [n_2^2 b^4 - (n_1 a^2 - n_3 c^2)^2], \tag{4}$$

$$R_3^3 = \frac{(\dot{abc})}{abc} + \frac{1}{2} \frac{1}{a^2 b^2 c^2} [n_3^2 c^4 - (n_1 a^2 - n_2 b^2)^2], \tag{5}$$

where $\dot{a} = da/dt$, etc. If the metric on M_4 is Ricci-flat, i.e. $R_b^a = 0$, Eqs. (2)–(5) are integrable in most cases. The resulting manifolds are singular.²

For non-Ricci-flat manifolds, in particular for manifolds with $R_b^a = \Lambda \delta_b^a$, $\Lambda < 0$, the Einstein equations are not in general integrable. However, a number of solutions with extra symmetries exist. For example, in Section 5 we discuss the Bergmann metric — a Bianchi type II solution with negative cosmological constant. Unlike the Ricci-flat Bianchi type II solution, the Bergmann metric is complete.

2.2. Monge–Ampère equation

From the theory of Kähler manifolds it is known that the Kähler metric may be obtained from a real-valued function of complex holomorphic coordinates $z, \bar{z} = \{z^a, \bar{z}^a\}$ called the Kähler potential:

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(z, \bar{z}).$$

Here $\partial_a = \partial/\partial z^a$ and $\partial_{\bar{b}} = \partial/\partial \bar{z}^b$. A Kähler manifold is Einstein–Kähler if the Kähler metric $g_{a\bar{b}}$ satisfies the Einstein equations:

$$\mathcal{R}_{a\bar{b}} = \Lambda g_{a\bar{b}}.$$

These are equivalent to the requirement that the Kähler potential satisfy the so-called Monge–Ampère equation obtained as follows. The Ricci tensor is given by

$$\mathcal{R}_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \log \det g(z, \bar{z}),$$

and hence the Einstein–Kähler condition reduces to

$$\det(\partial_a \partial_{\bar{b}} K) = e^{-\Lambda K}. \tag{6}$$

In general this is a complex partial differential equation solving which is not straightforward. If, however, the manifold possesses certain amount of symmetry, the Monge–Ampère equation may reduce to an ODE. In the following sections we shall deduce the Monge–Ampère equations and their solutions for most of the Kähler manifolds that we study.

3. Vacuum solutions of Bianchi type I and II

We shall begin by assuming that the domain walls are invariant under three translations, i.e. that they are Bianchi type I, or Kasner, but we shall find that to be supersymmetric

² Many self-dual four-dimensional vacuum solutions of various Bianchi types have been found in [12].

objects they should instead be invariant under the nilpotent Bianchi type II group Nil often called the Heisenberg group.

3.1. Kasner walls

One’s first idea might be to choose the metric $g_{\alpha\beta}$ on M_4 to depend only on one “transverse” coordinate, call it t , and to be independent of the other three coordinates x^1, x^2, x^3 , say. Thus the metric would admit an isometric action of \mathbb{R}^3 or, if we identify, T^3 and falls into the vacuum Bianchi I or Kasner class of solutions [13]:

$$ds^2 = dt^2 + t^{2\alpha_1}(dx^1)^2 + t^{2\alpha_2}(dx^2)^2 + t^{2\alpha_3}(dx^3)^2, \tag{7}$$

where constants $\alpha_1, \alpha_2, \alpha_3$ satisfy

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2. \tag{8}$$

There are two problems with this metric. The first problem is that if metric (7) is not flat, it is singular at the domain wall $t = 0$; and the second problem is that it is not BPS.

Consider for example the rotationally symmetric case

$$(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{4}{3}, \frac{4}{3}, \frac{2}{3}\right).$$

Metric (7) is then singular at $t = 0$, but complete as $t \rightarrow \infty$. Thus, in accordance with our general remarks made in Section 1, it is not asymptotically flat in the usual sense although the curvature falls off as t^{-2} .

3.2. BPS walls: Bianchi type II

To be BPS the manifold M_4 must admit at least one, and hence at least two covariantly constant spinors. If the solution admits at least one tri-holomorphic Killing vector it may be cast in the form

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 + V d\mathbf{x}^2, \tag{9}$$

where $\mathbf{x} = (x^1, x^2, x^3)$ with

$$\text{curl } \omega = \text{grad } V.$$

One may either regard the ignorable coordinate τ as lying in the world-volume of the p -brane or as a Klauza–Klein coordinate. Obviously one may entertain both interpretations simultaneously in which case one is considering the double-dimensional reduction of a brane in a lower dimensional space [14]. For the time being we will not tie ourselves down on this point. In order to get a domain wall solution we want some sort of invariance under two further translations and we are naturally led to choose for the harmonic function

$$V = z,$$

where we now interpret the coordinate z as a transverse coordinate. With $x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$ metric (9) becomes

$$ds^2 = z dz^2 + z(dx^2 + dy^2) + z^{-1}(d\tau - x dy)^2. \tag{10}$$

The transverse proper distance is given by

$$t = \frac{2}{3}z^{3/2}.$$

The metric is complete as $z \rightarrow +\infty$, the curvature again falls off as t^{-2} but it clearly has a singularity at $z = 0$, at which the signature changes from $(+++)$ to $(---)$. We shall return to this point shortly.

The Monge–Ampère equation and the Kähler potential for this metric will be given in Section 5.3 where suitable complex coordinates are introduced.

3.3. Geometrical considerations and the Heisenberg group

Evidently metric (10) is not invariant under translations in the y coordinate; nevertheless it admits a three-dimensional group of isometries. The metric may be written in the general form (1) so that the group of isometric motions is manifest

$$dt^2 + \left(\frac{3t}{2}\right)^{-2/3} (\sigma^3)^2 + \left(\frac{3t}{2}\right)^{2/3} ((\sigma^1)^2 + (\sigma^2)^2), \tag{11}$$

where $\{\sigma^k\}$ are left-invariant one forms on the Nil or the Heisenberg group. From this point on we shall refer to the metric (10) (or (11)) as the *Heisenberg* metric. The Heisenberg group may be defined as the nilpotent group Nil = $\{g\}$ of 3×3 real-valued upper triangular matrices

$$g = \begin{pmatrix} 1 & x & \tau \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra of Nil has as a basis³:

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the only non-vanishing commutator is

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3.$$

The basis elements $\{\mathbf{e}_k\}$ correspond to three right-invariant Killing vector fields

$$\mathbf{R}_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial \tau}, \quad \mathbf{R}_2 = \frac{\partial}{\partial y}, \quad \mathbf{R}_3 = \frac{\partial}{\partial \tau},$$

³ We adhere to the conventions that if a group G with Lie algebra $[\mathbf{e}_a, \mathbf{e}_b] = C_a^c{}_b \mathbf{e}_c$ acts on the left on a manifold M then the Killing vector fields \mathbf{R}_a have Lie brackets $[\mathbf{R}_a, \mathbf{R}_b] = -C_a^c{}_b \mathbf{R}_c$, while the left-invariant one-forms $g^{-1} dg = \mathbf{e}_a \sigma^a$ satisfy $d\sigma^c = -\frac{1}{2} C_a^c{}_b \sigma^a \wedge \sigma^b$.

for which the only non-vanishing commutator is

$$[\mathbf{R}_1, \mathbf{R}_2] = -\mathbf{R}_3; \tag{12}$$

and three left-invariant one-forms

$$\sigma^1 = dx, \quad \sigma^2 = dy, \quad \sigma^3 = d\tau - x dy, \tag{13}$$

whence

$$d\sigma^3 = -\sigma^1 \wedge \sigma^2.$$

In addition this metric admits a rotational Killing vector of the form

$$\mathbf{m} = -x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x} - \frac{x^2 - y^2}{2} \frac{\partial}{\partial \tau}. \tag{14}$$

This Killing field \mathbf{m} induces a rotation of σ^1 into σ^2 but leaves σ^3 invariant.

The four-dimensional Heisenberg manifold (10) is Ricci-flat Kähler and hence carries a hyperKähler structure. To exhibit the three Kähler forms let us introduce the following orthonormal basis of one-forms $\{e^0, e^k\}$:

$$\begin{aligned} e^0 &= z^{-1/2}(d\tau - x dy), & e^1 &= z^{1/2} dx, \\ e^2 &= z^{1/2} dy, & e^3 &= z^{1/2} dz. \end{aligned} \tag{15}$$

In terms of these frames the three self-dual two-forms which are the Kähler forms are

$$\Omega_x = e^0 \wedge e^1 + e^2 \wedge e^3, \quad \text{and cyclic permutations,} \tag{16}$$

and for the Heisenberg metric (10) these become

$$\begin{aligned} \Omega_x &= (d\tau - x dy) \wedge dx + z dy \wedge dz, \\ \Omega_y &= (d\tau - x dy) \wedge dy + z dz \wedge dx, \\ \Omega_z &= (d\tau - x dy) \wedge dz + z dx \wedge dy. \end{aligned} \tag{17}$$

It is easily seen that the self-dual two-forms Ω_x, Ω_y and Ω_z — the Kähler forms — are closed and hence harmonic. They are clearly invariant under the action of the Heisenberg group. However, only Ω_z is invariant under the circle action generated by the rotational Killing field \mathbf{m} (14).

3.4. Circle bundles and volume growth

If one wishes to identify the coordinates x and y to obtain a two-dimensional torus one is forced to make appropriate identifications of the coordinate τ . The result is a circle bundle over T^2 . Such bundles M_k are indexed by an integer k which is essentially the Chern class. They are often referred to as *Nilmanifolds*.

If the periods of (x, y, τ) are (L_x, L_y, L_τ) then one must have

$$k = \frac{L_x L_y}{L_\tau} \in \mathbb{Z}. \tag{18}$$

If that is true then

$$\exp(L_x \mathbf{e}_1) \exp(L_y \mathbf{e}_2) \exp(-L_x \mathbf{e}_1) \exp(-L_y \mathbf{e}_2) = \exp(L_x L_y \mathbf{e}_3),$$

and hence $\exp(L_x \mathbf{e}_1)$, $\exp(L_y \mathbf{e}_2)$ and $\exp(L_\tau \mathbf{e}_3)$ will close on a discrete group \mathcal{N}_k . One then has

$$M_k = \text{Nil}/\mathcal{N}_k,$$

which clearly admits a global right action of $U(1) = \exp(z\mathbf{e}_3)$.

The curvature of the connection pulled back to the base T^2 is

$$F = d\sigma^3 = dy \wedge dx$$

and the Dirac quantization condition is

$$\frac{1}{L_\tau} \int_{T^2} F = k \in \mathbb{Z}. \tag{19}$$

We shall see later that the relevant value of the integer k in our case is $k = 3$. Formula (19) takes on a more conventional appearance if one chooses $L_\tau = 2\pi$. Alternatively, one could think of $e = 2\pi/L_\tau$ as an electric charge. It has been known for some time that a Kaluza–Klein reduction on the Heisenberg group gives rise to a uniform magnetic field [15]. Interestingly, since the present solution is BPS, it should be stable against production of monopole–anti-monopole pairs. This is in contrast to other examples of magnetic fields in Kaluza–Klein theories, for example in vacua studied in [16,17] such monopole–anti-monopole pairs are produced.

The curves of constant (x, y, τ) are geodesics orthogonal to the group orbits and the coordinate t is the radial distance. If the orbits are compact we may estimate how the four-volume of a geodesic ball increases with t by calculating the four-volume of the metric (10) between $t = t_1$ say and t . This is easily seen to grow with t as $t^{4/3}$. We shall use this fact in Section 4.5 to compare with the work of Bando, Kobayashi, Tian and Yau [6–8] on an exact metric on the complement of a smooth cubic curve in $\mathbb{C}\mathbb{P}^2$.

4. Resolution of the singularity

In this section we describe the physical motivation for resolving the singularity of the Heisenberg manifold (10) and analyse the underlying mathematical structure of the proposed resolution.

4.1. 8-branes, 6-branes and T-duality

The metric (10) has been reached previously by a different route. The massive type IIA 10-dimensional theory of Romans [18] admits BPS solutions corresponding to Dirichlet 8-branes whose properties have been discussed by Polchinski and Witten [19] and Bergshoeff et al. [20]. The solutions are based on a harmonic function of the coordinate transverse to the

8-branes which has discontinuities at the location of the 8-branes. The relation of Romans' theory to 11-dimensional supergravity theory is unclear.⁴ However, under a double T-duality with respect to two coordinates lying in the 8-brane, x and y say, it may be reduced to a 6-brane solution of the IIA theory compactified to eight dimensions. Under T-duality, the coordinates x and y become transverse coordinates, and strictly speaking, because the solution is independent of the coordinates x and y one has a superposition of 6-branes. A 6-brane solution of the 10-dimensional type IIA theory may be lifted to 11-dimensions to give a BPS 7-brane wrapped around the 11th dimension. In other words the 11-dimensional 7-brane is a Ricci-flat metric of the form

$$\mathbb{E}^{6,1} \times M_4, \tag{20}$$

where M_4 is a multi-Taub–NUT metric of the form (9):

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 + V d\mathbf{x}^2$$

with

$$\text{curl } \omega = \text{grad } V.$$

The coordinate τ is the 11th direction. Coordinates \mathbf{x} are transverse to the 6-brane. A single 6-brane corresponds to the Taub–NUT metric with positive mass which has

$$V = 1 + \frac{1}{r}.$$

In order to get a superposition of 6-branes which is independent of x and y (at least up to gauge transformations) one should choose

$$V = z,$$

and this is indeed what Bergshoeff et al. [20] found.

4.2. Sources

As it stands, metric (10) is singular at $z = 0$. In fact this singularity resembles the singularity in the self-dual Taub–NUT metric with negative mass parameter for which $V = 1 - 1/r$ in (9). On the three-surface $r = 1$ the metric changes signature from $(++++)$ to $(----)$. The Taub–NUT metric with negative mass parameter is known to be asymptotic to a complete topologically non-singular self-dual Riemannian manifold called the Atiyah–Hitchin manifold [22]. The presence of the singularity at $r = 1$ is a clear indication of the fact that the Taub–NUT approximation is broken already at values of r greater than 1. It is natural to suppose that something similar may be happening in the case of the Heisenberg metric (10). Indeed in the next section we shall make a concrete proposal for the exact metric.

⁴ See, however, a very recent paper of Hull [21].

However, Bergshoeff et al. [20] and others writing on supergravity domain walls [23] do something else. They replace z by $|z|$ which results in a configuration symmetric under the reflection $z \rightarrow -z$. The justification for this procedure is that one has inserted a distributional source at $z = 0$ representing the domain wall and the regions $z > 0$ and $z < 0$ correspond to the two sides of the domain wall. Geometrically this resembles but is not equivalent to the procedure of Israel [24] used in classical general relativity who describes a shell of matter by gluing together two smooth space–times M^\pm across a hypersurface Σ . The Israel matching conditions are that the two metrics g_{ij}^\pm induced on Σ from M^\pm agree. One then evaluates the distributional stress tensor from the discontinuity in the second fundamental forms $(K_{ij}^+ - K_{ij}^-)$ across Σ .

From the point of view of M-theory there are two objections to doing this in the present case:

- There are no obvious sources in M-theory.
- The induced metric on the hypersurface Σ given by $z = 0$ is singular.

4.3. Orbifold walls

An alternative attitude to the singularity of (10) at $z = 0$ would be to identify the region $z > 0$ with the region $z < 0$. The singularity would then be viewed as a consequence of the fact that the reflection has a fixed point set. Thus one has something analogous to the two orbifold domain walls at the ends of an interval in Hořava and Witten’s compactification of the 11-dimensional M-theory on $S^1 \times \mathbb{Z}_2$ to give the $E_8 \times E_8$ heterotic theory in 10 dimensions [2,3].⁵ In the formulation of Lukas et al. [26] one considers the 11-dimensional metric on $\mathbb{E}^{3,1} \times S^1/\mathbb{Z}^2 \times X^6$

$$ds^2 = \frac{1}{H} g_{\mu\nu}^4 + H^2 dy^2 + H g_{AB}^6,$$

where $g_{\mu\nu}^4$ is the four-metric on the flat Minkowski space–time $\mathbb{E}^{3,1}$, y the coordinate on the interval S^1/\mathbb{Z}^2 ranging from $-\pi\rho$ to $\pi\rho$, and g_{AB}^6 is the metric on the compact Calabi–Yau space X^6 . Function H is a harmonic function linear in y and invariant under the reflection $y \rightarrow -y$. In addition, there is a non-vanishing four-form field strength in the 11-dimensional theory. In the effective five-dimensional theory obtained by generalized Kaluza–Klein dimensional reduction on the Calabi–Yau space X^6 , this solution can be viewed as a pair of 3-brane domain walls on the orbifold fixed planes $y = 0$ and $y = \pi\rho$. The 3-branes are in fact the M-theory 5-branes with two world-volume dimensions “wrapped” on a two-cycle in X^6 .

In our case the solution is defined on $\mathbb{E}^{6,1} \times \mathbb{R}_+ \times \text{Nil}$, where \mathbb{R}_+ is parametrized by $z > 0$ and the three-manifold Nil parametrized by $\{x, y, \tau\}$ is the group manifold of the Heisenberg group. We may think of this as a 9-brane solution of 11-dimensional supergravity where the world-volume of the 9-brane is taken to be $\text{Nil} \times \mathbb{E}^{6,1}$. Replacing Nil by $M_k = \text{Nil}/\mathcal{N}_k$ defined in Section 3.4 amounts to “wrapping” the 9-brane on the S^1 bundle over T^2 . Just

⁵ For an earlier discussion of gauge theories on orbifolds see e.g. [25].

as in the Hořava–Witten case we do not have the full $SO(9, 1)$ Lorentz invariance, rather it is broken to $SO(6, 1) \times \text{Nil}$.

4.4. Scherk–Schwarz reduction to seven dimensions

In the light of the comments above, particularly the absence of Lorentz invariance, perhaps the most attractive interpretation of the solution (10) is that adopted by Lavrinenko et al. [27]. One regards it as a solution of the so-called “massive” eight-dimensional theory that is obtained by reducing 11-dimensional supergravity *à la* Scherk and Schwarz [28]. In other words, one restricts the 11-dimensional theory to solutions invariant under the action of the three-dimensional Heisenberg group. The resulting theory has a potential for the scalar fields arising from the reduction and as a consequence there is no solution with the eight-dimensional Poincaré invariance. Lavrinenko et al. [27] therefore proposed using the BPS solution (10). In their interpretation z is, as with us, the transverse coordinate (i.e. the eighth coordinate) and (x, y, τ) are the ninth, 10th and 11th coordinates in no particular order. Since the size of the x and y directions goes to infinity as $z \rightarrow \infty$ and the size of the τ direction goes to zero, the Scherk–Schwarz reduction is not really a compactification even if one identifies the coordinates so as to obtain a circle bundle over a two-torus. It is, however, certainly a consistent truncation of the theory.

4.5. Blowing up the singularity

If the configuration (10) really does come from M-theory we still face the problem of the source. We have two possibilities:

- *either* to follow Bergshoeff et al. [20] and Lavrinenko et al. [27] and take the view that the domain wall has two sides,
- *or* to adopt the orbifold interpretation and identify the regions of positive and negative z .

Both approaches give rise to a singularity. The question arises as to whether one can somehow smooth out the singularity? We are going to argue that the answer is *no* if we adopt the first course and *yes* if we adopt the second. Assuming that only gravity with no extra form-fields is present, we thus seek a non-singular Ricci-flat BPS metric which is asymptotic to the Heisenberg metric (10).

To see that the first approach is ruled out we note that if the singularity could be resolved, then keeping coordinates (x, y, τ) would give a complete Ricci-flat metric on $\mathbb{R} \times \Sigma$, where Σ is a closed complete three-manifold. In particular, the manifold would have two “ends”, i.e. two infinite regions. However if this were true we could construct a “line” between the two ends, i.e. a geodesic which minimizes the length between any two points lying on it. But by the Cheeger–Gromoll theorem this is impossible (see e.g. [29]). Thus we are forced to adopt the second course of action which is investigated in detail in the following section.

Before doing so it is perhaps worthwhile pointing out the analogy of the situation in question with the case of the blow up of $\mathbb{E}^4/\mathbb{Z}_2$. One might have wondered if it is possible to glue together two copies of \mathbb{E}^4 to get a Ricci-flat wormhole-like structure with topology $\mathbb{R} \times \mathbb{R}\mathbb{P}^3$. Again by the Cheeger–Gromoll theorem this cannot happen. In fact we know that

the correct blow up of $\mathbb{E}^4/\mathbb{Z}_2$ is the Eguchi–Hanson manifold on the cotangent bundle of $\mathbb{C}\mathbb{P}^1$ [30] and that this manifold has only one infinite region.

Another completely analogous situation is the Taub–NUT approximation to an orientifold plane. This is obtained by taking the metric (9) with $V = 1 - 1/r$ and making a further identification [31]. The metric is incomplete because of the singularity at $r = 1$. This singularity cannot be resolved by joining together two copies of the Taub–NUT metric across $r = 1$ because this would also produce a manifold with two ends. The correct way to blow up the singularity of the Taub–NUT metric is to pass to the Atiyah–Hitchin manifold.

4.6. Complement of a cubic in $\mathbb{C}\mathbb{P}^2$: the BKTY metric

We now turn to the problem of finding, or more properly speaking identifying, the exact metric of which the Heisenberg metric (10) is an asymptotic approximation. This task is greatly facilitated by the extremely helpful review of Kobayashi [32] on degenerations of the metric on K3 and in what follows we shall rely heavily on that reference. The general self-dual four-metric on K3 has (including an overall scale) a 58 parameter moduli space. As we move to the boundary of the moduli space in certain directions the four-metric may decompactify, while remaining complete and non-singular. Among the degenerations discussed by Kobayashi there is one he refers to as type II. It may be constructed by considering the complement $M_4 = \mathbb{C}\mathbb{P}^2 \setminus C$ of a smooth cubic curve C in the complex plane $\mathbb{C}\mathbb{P}^2$. This has a Kähler metric: the Fubini–Study metric which is incomplete because the cubic has been removed. However, using general existence theorems for solutions of the Monge–Ampère equation Yau, Tian, Bando and Kobayashi [6–8] have shown that there exists a complete non-singular Ricci-flat Kähler (and hence self-dual) metric on M_4 . Clearly the metric must blow up on the cubic C which corresponds to infinity.

Consider now the neighbourhood of the cubic C . The curve itself is topologically a two-torus T^2 . A normal neighbourhood consists of a disc bundle over T^2 . The centre of the disc corresponds to infinity in M_4 . The radial direction corresponds to a geodesic in the self-dual metric. A surface of constant radius is a circle bundle over the torus. This is the three-dimensional Nilmanifold.

Kobayashi tells us that the Nilmanifold collapses as we approach infinity in such a way that the metric spheres are an S^1 -bundle over T^2 , the size of the S^1 falls off as $t^{-1/3}$ (t is the radial distance) and the size of each cycle in T^2 grows as $t^{1/3}$. The volume of a metric ball grows as $t^{4/3}$. This is exactly the behaviour of the Heisenberg metric (11). It is therefore very plausible that the metrics constructed by Yau, Tian, Bando and Kobayashi do indeed asymptote to the metric (11). In what follows we shall assume that this is true.

The topology of M_4 is non-trivial⁶: it is not simply connected and has

$$H_1(M_4) = \mathbb{Z}_3, \quad H_2(M_4) = \mathbb{Z} \oplus \mathbb{Z}.$$

Hence if arguments like those in [33] apply, the manifold should admit at least two normalizable anti-self-dual two-forms. Using the analysis of Hawking and Pope [33] one deduces

⁶ We thank Ryushi Goto for this computation.

that there should be a $2 \times 3 = 6$ dimensional family of transverse traceless zero modes of the Lichnerowicz operator. Adding the trivial overall scaling we expect to find a seven-dimensional family of metrics.

4.7. Gravitational action

Complete Ricci-flat (vacuum) Einstein manifolds are gravitational instantons. It is of interest to estimate their contribution to the path integral of the Euclidean quantum gravity by evaluating their gravitational action. If \mathcal{M} is a non-compact manifold or a compact manifold with boundary $\partial\mathcal{M}$ the gravitational action is

$$- \frac{1}{16\pi} \int_{\mathcal{M}} R - \frac{1}{8\pi} \int_{\partial\mathcal{M}} \text{Tr}\mathcal{K}, \tag{21}$$

where R is the Ricci tensor and \mathcal{K} is the second fundamental form on \mathcal{M} . The first term is the contribution from the bulk which vanishes for Ricci-flat manifolds; the second term is the contribution from the boundary (possibly boundary at infinity). Traditionally only four-manifolds were regarded as gravitational instantons, however expression (21) is valid for complete Ricci-flat Einstein manifolds in any dimension.

Let us estimate the contribution from the boundary. Let \mathbf{n} be a vector normal to the boundary $\partial\mathcal{M}$, then the second term in (21) is

$$\frac{1}{8\pi} \int_{\partial\mathcal{M}} \text{Tr}\mathcal{K} = \frac{1}{8\pi} \frac{\partial}{\partial \mathbf{n}} (\text{Vol } \partial\mathcal{M}).$$

By $\text{Vol } \partial\mathcal{M}$ we mean the unit volume of the boundary. For four-dimensional Ricci-flat manifolds, if t is the radial distance the boundary term contribution to the action is finite if $\text{Vol } \partial\mathcal{M}$ be no faster than linear in t . This implies that the volume growth of a large metric sphere at infinity should grow no faster than t^2 . Similarly, for higher-dimensional instantons the “critical” volume growth for which the boundary contribution to the action is finite (but not necessarily vanishing) is t^2 .

The BKTY manifold possesses a non-compact complete Ricci-flat Kähler metric with the Heisenberg end and can thus be viewed as a gravitational instanton. Since it is Ricci-flat, its gravitational action receives no contribution from the first term in (21). At infinity the BKTY metric looks like the Heisenberg metric (11). The boundary of (11) at large values of t looks like an S^1 bundle over T^2 , where the two-torus parametrized by x, y , and τ is the fibre coordinate. Hence the second term in (21) is

$$\frac{1}{8\pi} \int_{\partial\mathcal{M}} \text{Tr}\mathcal{K} = \frac{1}{8\pi} \frac{\partial}{\partial \mathbf{n}} (\text{Vol } \partial\mathcal{M}) = \frac{1}{8\pi} \frac{\partial}{\partial t} \left[\left(\frac{3}{2}t\right)^{1/3} \mathcal{V} \right].$$

$\mathcal{V} = L_x L_y L_\tau$ where L_x, L_y and L_τ are the periods of x, y and τ as described in Section 3.4 and we get

$$\left(\frac{3}{2}t\right)^{-2/3} \frac{L_x L_y L_\tau}{16\pi}.$$

Note that the boundary unit volume grows as $t^{1/3}$ which is slower than the critical estimate t , hence it is not surprising that the gravitational action of the BKTY instanton is finite and tends to zero as $t \rightarrow +\infty$.

5. Bianchi types I and II with negative cosmological constant

In this section we look for p -brane solutions of the form $M_4 \times \mathbb{E}^{p-3,1}$, where now M_4 is not a Ricci-flat manifold but rather a four-manifold with negative cosmological constant. Such solutions may be interpreted as branes in the Anti-de Sitter background and are likely to be of interest in connection with the AdS/CFT [9–11] correspondence.

Here we focus on four-manifolds of Bianchi types I and II. The relevant metrics must be solutions of the Einstein’s equations (2)–(5) with $R_b^a = \Lambda \delta_b^a$, $\Lambda < 0$, and appropriate values of the structure constants n_k given in Table 1.

5.1. Bianchi type I

In this case $n_k = 0$ and Einstein equations are integrable. While there is no polynomial solution like the Kasner metric (7), the solution is obtained by replacing t^{α_k} in (7) with

$$\left(\frac{\sinh(\sqrt{-3\Lambda}t)}{\tanh((\sqrt{-3\Lambda}/2)t)} \right)^{1/3} \left(\tanh\left(\frac{\sqrt{-3\Lambda}}{2}t\right) \right)^{\alpha_k},$$

where the powers α_k again satisfy Eq. (8). Setting $\alpha_1 = 1$, we get a complete non-singular (in contrast with the singular Kasner metric) instanton if the coordinate x is suitably identified. This Kasner–Anti-de Sitter metric could be used to construct domain walls. Note, however, that like its vacuum counterpart (7) this metric is not BPS.

5.2. Bianchi type II: the Bergmann metric

Substituting the relevant structure constants into the Einstein equations (2)–(5) we find $b(t) = c_0 a(t) + c_1$. However, we are only interested in self-dual metrics (self-dual four-metrics are hyperKähler) and for these the constants c_0, c_1 are $c_0 = 1, c_1 = 0$. Hence we necessarily have $b(t) = a(t)$ and the Einstein equations reduce to

$$-- \Lambda = 2 \frac{\ddot{a}}{a} + \frac{\ddot{c}}{c}, \tag{22}$$

$$-- \Lambda = \frac{(\dot{a}a\dot{c})}{a^2c} - \frac{c^2}{2a^4}, \tag{23}$$

$$-- \Lambda = \frac{(\dot{c}a^2)}{a^2c} + \frac{c^2}{2a^4}. \tag{24}$$

It is not straightforward to solve Eqs. (22)–(24), in fact it is not clear whether they are integrable in general. There exists, however, a special solution for which

$$a(t) = Ae^{\alpha t}, \quad c(t) = Be^{\gamma t},$$

where α, γ, A, B are constants. Substituting this ansatz into (22)–(24) we find

$$\alpha^2 = -\frac{\Lambda}{6} = \frac{B^2}{4A^4}, \quad \gamma = 2\alpha.$$

Since only the ratio A/B plays a role we are free to choose $A = 1$, giving $B = \sqrt{-2\Lambda/3}$. Clearly, the solution is invariant under time-reversal $t \rightarrow -t$; it therefore suffices to consider negative parameters $\alpha < 0$. The Bianchi type II four-metric thus becomes

$$ds^2 = dt^2 + e^{-2t\sqrt{-\Lambda/6}}((\sigma^1)^2 + (\sigma^2)^2) + \sqrt{-\frac{2\Lambda}{3}}e^{-4t\sqrt{-\Lambda/6}}(\sigma^3)^2, \tag{25}$$

where $\{\sigma^k\}$ are the left-invariant one-forms (13). A convenient choice of the cosmological constant is $\Lambda = -2/3$, for which (25) becomes

$$ds^2 = dt^2 + e^{-t}((\sigma^1)^2 + (\sigma^2)^2) + e^{-2t}(\sigma^3)^2. \tag{26}$$

In (26) one recognizes the Bergmann metric — a non-compact Kähler symmetric space $SU(2, 1)/U(2)$. Note that the Bergmann metric is complete (unlike the vacuum Bianchi type II metric (10)). It is also BPS.

Another form of the Bergmann metric where the $U(2)$ group action is manifest is

$$ds^2 = \frac{dR^2}{(1 + (\Lambda/6)R^2)^2} + \frac{R^2}{4(1 + (\Lambda/6)R^2)^2}(\sigma^3)^2 + \frac{R^2}{4(1 + (\Lambda/6)R^2)}((\sigma^1)^2 + (\sigma^2)^2). \tag{27}$$

Expressed in this form the metric is a limiting case of the general $U(2)$ -invariant Taub–NUT–Anti-de Sitter family of metrics [34] when $\mathcal{N} \rightarrow \infty$ (\mathcal{N} is the NUT charge). Such a family can be written in the form (see Eq. (2.6) in [34]):

$$\frac{U(r)}{f(r)} dr^2 + 4\mathcal{N}^2 \frac{f(r)}{U(r)} (\sigma^3)^2 + r^2 U(r) ((\sigma^1)^2 + (\sigma^2)^2), \tag{28}$$

where $f(r) = 1 + (\Lambda/3)r^2(1 + 4\mathcal{N}/r)$ and $U(r) = 1 + 2\mathcal{N}/r$. Now, to take the limit of large NUT charge, rescale the radial coordinate $r = \rho/\mathcal{N}$ in the above formulae and take $\mathcal{N} \rightarrow \infty$. Metric (28) becomes

$$ds^2 = \frac{4 d\rho^2}{2\rho(1 + (4\Lambda/3)\rho)} + 2\rho \left(1 + \frac{4\Lambda}{3}\rho \right) (\sigma^3)^2 + 2\rho((\sigma^1)^2 + (\sigma^2)^2), \tag{29}$$

and taking $2\rho = R^2/4(1 + (\Lambda/6)R^2)$ we get back to expression (27).

5.3. Horospheres

To elucidate the geometrical structure of the Bergmann manifold and the role played by the Heisenberg group we shall now describe the way the Bergmann manifold arises as the set of horospheres of an odd-dimensional Anti-de Sitter space. We shall also obtain the

Monge–Ampère equation for a Bianchi type II four-manifold. A solution to the equation with a negative cosmological constant gives the Kähler potential for the Bergmann metric. We make use of the defined complex coordinates and solve the Ricci-flat Monge–Ampère equation to obtain the Kähler potential for the Heisenberg metric (11).

Suppose G/H is a non-compact Riemannian symmetric space. The Iwasawa theorem (see e.g. [35]) tells us that every element $g \in G$ may be uniquely expressed as

$$g = han,$$

where $h \in H \subset G$, $a \in A \subset G$, $n \in N \subset G$; A is an abelian and N is the nilpotent subgroup of G . This means that we may also think of G/H as a solvable group $G_{\text{solv}} = A \ltimes N$ with a left-invariant metric, where \ltimes denotes a semi-direct product and $A \ltimes N$ is a solvable group. The orbits of N in G/H are called *horospheres*. The set of horospheres is labelled by elements of A . They are permuted by elements of H . The simplest example would be an n -dimensional real hyperbolic space $G/H = \mathcal{H}^n$, $A = \mathbb{R}_+$ and $N = \mathbb{R}^{n-1}$. If we think of \mathcal{H}^n as a quadric in $\mathbb{E}^{n,1}$ ($(n + 1)$ -dimensional Minkowski space–time) then the horospheres are the intersections of the quadric with a family of parallel null hypersurfaces related by boosts. There is a similar description for an n -dimensional Anti-de Sitter space AdS_n regarded as a quadric in $\mathbb{E}^{n-1,2}$. This description of the hyperbolic and the Anti-de Sitter spaces will be useful in Section 8.

The case of a complex hyperbolic n -space $\mathcal{H}^n_{\mathbb{C}}$ is slightly more complicated. Thinking of $\mathbb{E}^{2n,2}$ as $\mathbb{C}^{n,1}$, the $(2n + 1)$ -dimensional Anti-de Sitter space, AdS_{2n+1} , is given by the quadric

$$|z^0|^2 - \sum_{a=1}^n |z^a|^2 = 1.$$

Then the complex hyperbolic n -space $\mathcal{H}^n_{\mathbb{C}}$ is obtained by identifying z^a with $e^{i\theta} z^a$, $a = 1, \dots, n$. Thus (z^0, \dots, z^n) are homogeneous coordinates on $\mathcal{H}^n_{\mathbb{C}}$. The nilpotent group N turns out to be the $(2n + 1)$ -dimensional Heisenberg group (see Section 7). Let us see how this works in detail. It is helpful to recall that the inhomogeneous coordinates ζ^a are defined in the usual way as $\zeta^a = z^a/z^0$ and make manifest the action of $U(n)$ on $\mathcal{H}^n_{\mathbb{C}}$. Our aim is to find a set of coordinates to make the action of the Heisenberg group N manifest.

Let us first introduce complex null coordinates u and v :

$$u = \frac{1}{\sqrt{2}}(z^0 + z^n), \quad v = \frac{1}{\sqrt{2}}(z^0 - z^n).$$

Define z and w^i , $i = 1, \dots, n - 1$ to be

$$z = \frac{u}{v}, \quad w^i = \frac{z^i}{v}.$$

In terms of the inhomogeneous coordinates $\{\zeta^i, \zeta^n\}$ these are

$$z = \frac{2}{1 - \zeta^n} - 1, \quad w^i = \frac{\sqrt{2}\zeta^i}{1 - \zeta^n}. \tag{30}$$

A complex hyperbolic n -space is topologically the interior of a unit ball in \mathbb{C}^n and the map $(\zeta^i, \zeta^n) \rightarrow (w^i, z)$ provides a bi-holomorphism from the interior of the unit ball in \mathbb{C}^n into the interior of the paraboloid

$$z + \bar{z} > \sum_i |w^i|^2.$$

As we have mentioned above, $\mathcal{H}_{\mathbb{C}}^n$ is obtained from AdS_{2n+1} as the base of the Hopf fibration, with the time-like Hopf fibre parametrized by θ such that

$$\begin{aligned} u &= \frac{z^0 + z^n}{\sqrt{2}} = \frac{ze^{i\theta}}{(z + \bar{z} - \sum_i |w^i|^2)^{1/2}}, \\ v &= \frac{z^0 - z^n}{\sqrt{2}} = \frac{e^{i\theta}}{(z + \bar{z} - \sum_i |w^i|^2)^{1/2}}, \\ z^i &= \frac{w^i e^{i\theta}}{(z + \bar{z} - \sum_i |w^i|^2)^{1/2}}. \end{aligned} \tag{31}$$

The real quantity $(z + \bar{z} - \sum_i |w^i|^2)$ is invariant under the action of the Heisenberg group N parametrized by (a^i, b) :

$$\begin{aligned} w^i &\rightarrow w^i + a^i, \\ z &\rightarrow z + ib + \sum_i \frac{1}{2} |a^i|^2 + \bar{a}^i w^i. \end{aligned} \tag{32}$$

Considered as a subgroup of $SU(n, 1) \subset SO(2n, 2)$, N acts on $\mathcal{H}_{\mathbb{C}}^n$ as

$$\begin{aligned} u &\rightarrow u + \left(ib + \sum_i \frac{1}{2} |a^i|^2 \right) v + \bar{a}^i z^i, \\ v &\rightarrow v, \\ z^i &\rightarrow z^i + a^i. \end{aligned}$$

Finally, the abelian group $A = \mathbb{R}_+$ parametrized by λ acts as $(z, w^i) \rightarrow (\lambda^2 z, \lambda w^i)$ or

$$\begin{aligned} u &\rightarrow \lambda u, \\ v &\rightarrow \lambda^{-1} v, \\ z^i &\rightarrow z^i. \end{aligned}$$

Having identified all group actions clearly, let us now formulate the problem in terms of the Kähler potential. The Kähler potential for the metric on the horospheres may be obtained from the Kähler potential on the AdS_{2n+1} manifold which in terms of inhomogeneous coordinates ζ^a is given by

$$K(\zeta^a, \bar{\zeta}^a) = -\log \left(\sum_{i=1}^{n-1} |\zeta^i|^2 + |\zeta^n|^2 - 1 \right).$$

From (30),

$$\sum_i |\zeta^i|^2 + |\zeta^n|^2 - 1 = 2 \frac{|w^i|^2 - (z + \bar{z})}{(z + 1)(\bar{z} + 1)},$$

and hence the Kähler potential on the horospheres becomes (up to a Kähler gauge transformation)

$$K = -\log \left(z + \bar{z} - \sum_i |w^i|^2 \right). \tag{33}$$

Let us derive the Monge–Ampère equation to which Kähler potential (33) is a solution. Since the resulting metric contains the higher-dimensional extension of the Heisenberg group (see Section 7) as a group of isometries we must assume that the Kähler potential depends only on the real quantity $f = z + \bar{z} - \sum_i |w^i|^2$, the Monge–Ampère equation (6) becomes an ordinary differential equation:

$$(K')^{n-1} K'' = (-1)^{n-1} e^{-\Lambda K}, \tag{34}$$

where $K' = dK/df$.

Let us make the connection with the form of the four-dimensional Bergmann metric (26). In this case $n = 2$ and there are two complex coordinates (z, w) . We can pass from this parametrization to the parametrization of (26) in terms of (t, x, y, τ) as follows:

$$\begin{aligned} z - \bar{z} &= i \left(\tau - \frac{xy}{2} \right), & z + \bar{z} - w\bar{w} &= f, \\ w &= \frac{1}{2}(x + iy), & t &= e^f. \end{aligned} \tag{35}$$

For $n = 2$ the Monge–Ampère equation (34) becomes

$$K' K'' = -e^{-\Lambda K}. \tag{36}$$

In terms of the Kähler potential $K(f)$ the compatible Kähler metric is

$$ds^2 = -K' dw \wedge d\bar{w} + K''(dz - \bar{w} dw)(d\bar{z} - w d\bar{w}). \tag{37}$$

The Kähler potential (33) for $n = 2$:

$$K = -\log f = -\log \log t \tag{38}$$

is clearly a solution of Eq. (36) for $\Lambda = -3$. The Kähler metric $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$ — the Bergmann metric in complex coordinates — obtained from the Kähler potential (38) is

$$ds^2 = \frac{1}{f} dw \wedge d\bar{w} + \frac{1}{f^2} (dz - \bar{w} dw)(d\bar{z} - w d\bar{w}).$$

Rewriting this metric using definitions (35) we get the Bergmann metric in the standard form (26).

Let us solve the Monge–Ampère equation (36) in the Ricci-flat case $\Lambda = 0$. Here we present the four-dimensional case $n = 2$ leaving the treatment of the higher-dimensional example to Section 7. The solution of (36) should yield the Kähler potential for the Heisenberg metric (10). Integrating equation

$$K'K'' = -1,$$

we get

$$K' = \sqrt{c - 2f}, \quad K'' = -\frac{1}{\sqrt{c - 2f}},$$

where c is the integration constant which, without loss of generality, we may set to zero. Substituting these expressions into (37) and ignoring an overall constant factor we obtain the Heisenberg metric (11). Note that the preferred complex structure with respect to which the Kähler potential is defined is the one whose associated Kähler form is $U(1)$ -invariant. It is the two-form Ω_z (17) presented in Section 3.3.

6. Exotic asymptotics: Bianchi types VII₀ and VI₀

In this section we propose to investigate p -brane solutions whose asymptotics are more unusual than the ones considered in Section 3. We turn to Bianchi types VII₀ and VI₀ whose groups of isometric motions are $E(2)$ and $E(1, 1)$, respectively (see Table 1).⁷

We do not discuss the most general manifolds of the above types but rather focus on self-dual metrics (which are hyperKähler and hence BPS). Vacuum four-metrics of this kind are Ricci-flat Kähler metrics. The Einstein equations (2)–(5) in the self-dual case reduce to a set of first-order ODEs:

$$\frac{2}{a}a' = -n_1a^2 + n_2b^2 + n_3c^2, \tag{39}$$

$$\frac{2}{b}b' = -n_2b^2 + n_3c^2 + n_1a^2, \tag{40}$$

$$\frac{2}{c}c' = -n_3c^2 + n_1a^2 + n_2b^2. \tag{41}$$

For convenience we have introduced another radial coordinate η in place of t such that $dt = abc \, d\eta$ and $()'$ denotes differentiation with respect to η .

In Sections 6.1 and 6.2 we solve Eqs. (39)–(41) to obtain self-dual vacuum Bianchi type VII₀ and VI₀ metrics and discuss their properties. In Section 6.3 we discuss Bianchi type VII₀ and VI₀ manifolds with negative cosmological constant.

⁷ We consider type VII₀ spaces before type VI₀ spaces because the type VII₀ metric is in some sense simpler since its isometry group is Euclidean.

6.1. Vacuum solutions of Bianchi type VII₀ and Solvmanifolds

The group of isometries of a self-dual Bianchi type VII₀ metric is $E(2)$, whose structure constants are $n_1 = n_2 = 1$ and $n_3 = 0$ and a set of left-invariant one-forms is

$$\sigma^1 = \cos \tau dx + \sin \tau dy, \quad \sigma^2 = -\sin \tau dx + \cos \tau dy, \quad \sigma^3 = d\tau. \quad (42)$$

The self-duality equations (39)–(41) become

$$\frac{2}{a}a' = -a^2 + b^2, \quad \frac{2}{b}b' = -b^2 + a^2, \quad \frac{2}{c}c' = a^2 + b^2.$$

These are easily solved to yield the metric

$$ds^2 = \frac{\lambda^2}{2} \sinh 2\eta (d\eta^2 + (\sigma^3)^2) + \coth \eta (\sigma^1)^2 + \tanh \eta (\sigma^2)^2, \quad (43)$$

where λ is the integration constant. Let us estimate the volume of a large metric ball as was done in Section 3 for the Heisenberg manifold. Introducing as before the effective radial coordinate t to return to the ansatz (1), we find that the metric volume grows as t^2 for large t . Interestingly, this is the predicted volume growth of another type of a degeneration of the K3 surface in the Kobayashi’s review [32]. In fact Kobayashi proved the existence and completeness of a gravitational instanton whose three-dimensional hypersurfaces $t = \text{const}$ represent a collapse of *Solvmanifolds*.⁸ The non-compact complete metric on a degeneration of the K3 surface is expected to have quadratic volume growth of large metric spheres and have as an asymptotic metric the standard flat metric on $\mathbb{C}^* \times \mathbb{C}^*$. It is not known explicitly.

The present situation parallels the one we have already encountered with the Heisenberg metric. The Heisenberg metric (10) is singular at the origin, but the singularity is resolved by passing to another self-dual metric, the BKTY gravitational instanton, whose asymptotic form is exponentially close to the Heisenberg metric. The singularity at the origin of the Bianchi type VII₀ metric (43) may be resolved by passing to a non-singular manifold, whose existence and completeness is guaranteed by the general theorem of Kobayashi [32].

As we have pointed out, the large metric spheres have quadratic volume growth for large t . According to the estimates in Section 4.7, this is the critical volume growth for which the boundary term contribution to the gravitational action is constant and finite.

Alternatively, metric (43) may be obtained by solving the Monge–Ampère equation (6) for a Ricci-flat Kähler metric with appropriate symmetries. If we assume that the Kähler potential K is independent of the imaginary parts of $z^1 = u^1 + iv^1$ and $z^2 = u^2 + iv^2$ we obtain a metric with two commuting holomorphic isometries. If one further assumes that K depends only on the combination $\sqrt{(u^1)^2 + (u^2)^2}$ one gains an extra $SO(2)$ isometric action. From the first glance the resulting metric appears to be invariant under the direct

⁸ Usually Bianchi type VI₀ group $E(1, 1)$ is referred to as Solv or Solvable group. It is clear, however, that it is the type VII₀ manifold that has the volume growth predicted by Kobayashi. Its associated isometry group $E(2)$ is also solvable.

product $SO(2) \times \mathbb{R}^2$ but in fact it turns out that the group of isometries is the semi-direct product $SO(2) \ltimes \mathbb{R}^2 \equiv E(2)$. Thus one obtains a Bianchi type VII₀ metric.

A short explicit calculation reveals that in polar coordinates (r, τ) :

$$u^1 = r \cos \tau, \quad u^2 = r \sin \tau,$$

the Kähler potential depends only on r and the metric becomes

$$ds^2 = K'' dr^2 + rK'(\sigma^3)^2 + \frac{K'}{r}((\sigma^1)^2 + (\sigma^2)^2) + \left(K'' - \frac{K'}{r}\right)(\sigma^1)^2. \quad (44)$$

where $K' = dK/dr$ and $\{\sigma^k\}$ are the left-invariant $E(2)$ one-forms given in (42). Then the Monge–Ampère equation reduces to an ODE first written down by Calabi [36]:

$$\frac{K''K'}{r} = e^{-\Lambda K}, \quad (45)$$

with Λ the cosmological constant. When $\Lambda = 0$, the vacuum case, Calabi found that

$$K'(r) = \sqrt{r^2 - a^2}, \quad (46)$$

where a is an integration constant. The metric (44) with (46) is precisely the metric (43). Incidentally, this metric is the helicoid metric found by Aliev et al. [37] who obtained it using the connection between the real Monge–Ampère equation and minimal surfaces. It is singular at $\eta = 0$.

6.2. Vacuum solutions of Bianchi type VI₀

The group of motions preserving Bianchi type VI₀ metrics is $E(1, 1)$. From Table 1 the structure constants are $n_1 = 1$, $n_2 = -1$ and $n_3 = 0$, and hence the left-invariant one-forms are

$$\sigma^1 = \cosh \tau dx + \sinh \tau dy, \quad \sigma^2 = \sinh \tau dx + \cosh \tau dy, \quad \sigma^3 = d\tau. \quad (47)$$

The self-duality equations (39)–(41) become

$$\frac{2}{a}a' = -(a^2 + b^2), \quad \frac{2}{b}b' = a^2 + b^2, \quad \frac{2}{c}c' = a^2 - b^2.$$

These can be easily solved to give the following Ricci-flat Kähler metric:

$$ds^2 = \frac{\lambda^2}{2} \sin 2\eta(d\eta^2 + (\sigma^3)^2) + \cot \eta(\sigma^1)^2 + \tan \eta(\sigma^2)^2, \quad (48)$$

where λ is the integration constant. This metric is incomplete at the origin $\eta = 0$.

Such self-dual metrics of Bianchi type VI₀ were also displayed by Aliev et al. [37] and were referred to as catenoid metrics.

One can find a description of the metric (48) in terms of a Kähler potential as was done in the previous section for the type VII₀ metric (43). Again assuming that the Kähler potential K is independent of the imaginary parts of $z^1 = u^1 + iv^1$ and $z^2 = u^2 + iv^2$, we obtain a

metric with two commuting holomorphic isometries. Now assume that K depends only on the combination $\sqrt{(u^1)^2 - (u^2)^2}$ thus gaining an $SO(1, 1)$ isometric action. The resulting metric has as its group of isometries the semi-direct product $SO(1, 1) \ltimes \mathbb{R}^2 \equiv E(1, 1)$.

Defining new coordinates (r, τ) as

$$u^1 = r \cosh \tau, \quad u^2 = r \sinh \tau,$$

we find that the Kähler potential depends only on r and the metric becomes

$$ds^2 = K'' dr^2 - rK'(\sigma^3)^2 + \frac{K'}{r}((\sigma^1)^2 + (\sigma^2)^2) + \left(K'' - \frac{K'}{r}\right)(\sigma^1)^2, \quad (49)$$

where $\{\sigma^k\}$ are the left-invariant $E(1, 1)$ one-forms (47). The Monge–Ampère equation in this case differs from Calabi’s equation (45) by a sign

$$\frac{K''K'}{r} = -e^{-\Lambda K}. \quad (50)$$

In the vacuum case $\Lambda = 0$ and the Monge–Ampère equation

$$\frac{K''K'}{r} = -1$$

is solved by

$$K'(r) = \sqrt{a^2 - r^2}, \quad (51)$$

where a is the integration constant. This is the metric (48).

6.3. Bianchi type VII₀ and VI₀ with negative cosmological constant

If the cosmological constant Λ is negative, Calabi [36] proved that there exists a solution of (45) giving a complete non-singular metric on \mathbb{R}^4 . Unlike the analogous metric of Bianchi type II (the Bergmann metric of Section 5.2), but like the Kasner–Anti-de Sitter metric of Section 5.1, this metric is not homogeneous.

Analogously, Calabi’s argument concerning the solution of Eq. (45) with negative cosmological constant is applicable to the Bianchi type VI₀ case. It may thus be argued that solutions of (50) exist, although the completeness of the metrics has to be demonstrated.

7. Higher-dimensional examples of domain walls

In this section we would like to give examples of domain walls of the form $M \times \mathbb{E}^{p-3,1}$ in 11 dimensions, where manifold M remains hypersurface homogeneous but has dimension higher than four. Firstly we describe Calabi–Yau manifolds which are the higher-dimensional generalizations of the BKTY instanton of Section 4.6. We find their asymptotic metrics by solving the vacuum Einstein equations in $2n$ dimensions. We then give particular examples of such asymptotic metrics which arise as extensions of the vacuum Bianchi type

II or Heisenberg metric (11) to higher dimensions. In addition we present three series of higher-dimensional metrics originating from four-metrics of other Bianchi types: the type I or Kasner (7), the type VII₀ (43) and type VI₀ (48) metrics. We do so by generalizing the relevant Monge–Ampère equations and arguing the existence of solutions which provide the Kähler potentials for the metrics in question. By the extensions of Bianchi type metrics from four to arbitrary number of dimensions we mean the following. The Bianchi type I isometry group \mathbb{R}^3 extended to $n + 1$ dimensions is simply \mathbb{R}^n . The Bianchi type II three-dimensional group of isometries given by the left-invariant one-forms (13) may be easily generalized to a $(2n + 1)$ -dimensional group parametrized by (x_i, y_i, τ) , where $i = 1, \dots, n$, with left-invariant one-forms:

$$\sigma_i^1 = dx_i, \quad \sigma_i^2 = dy_i, \quad \sigma^3 = d\tau - \sum_{i=1}^n x_i dy_i, \tag{52}$$

satisfying

$$d\sigma^3 = - \sum_i \sigma_i^1 \wedge \sigma_i^2.$$

The Bianchi type VI₀ and VII₀ groups are three-dimensional groups $E(1, 1)$ and $E(2)$, respectively. In higher dimensions these become $E(n - 1, 1)$ and $E(n)$, respectively.

7.1. Higher-dimensional BKTY metrics and their asymptotic forms

In this section we shall rely heavily on [32] in which Kobayashi proves the existence theorem for complete Ricci-flat Kähler metrics on $X - D$ with $c_1(X) = [D]$, where X is a Fano manifold⁹ and D is a complex codimension 1 hypersurface in X . Here $[D]$ is a Poincaré dual of D . From Yau’s solution to Calabi’s conjecture one may infer that D carries a Ricci-flat Kähler metric. Although the gravitational instanton is not known explicitly, Kobayashi provides some detailed information on the asymptotic form of the metric. It has the following properties.

Let $t(p)$ measure the distance from some fixed point in $X - D$ to a point $p \in X - D$. Then far away from the chosen fixed point, i.e. for large t , the metric spheres have a structure of an S^1 -bundle over D . The size of the fibre, with respect to the induced metric on the metric spheres, decays as $t^{-(n-1)/(n+1)}$, while the radius of the $(n - 1)$ complex-dimensional base grows as $t^1/n + 1$. From this one estimates the volume growth of a large metric ball to be

$$\text{Vol} \sim \int t^{2(n-1) \cdot 1/n+1} \cdot t^{-(n-1)/(n+1)} dt \sim t^{2n/n+1}. \tag{53}$$

In this section we shall use the above information to make an ansatz for the asymptotic form of the metric and to show that it is an exact solution of the vacuum Einstein equations. We find that although the solution is Ricci-flat and Kähler it is singular. In fact, it bears the same relation to the gravitational instantons of Kobayashi as does the Heisenberg metric to

⁹ X is Fano if it has an ample canonical bundle, or in other words, if its first Chern class is positive, $c_1 > 0$.

the BKTY gravitational instanton. Outside a compact set the complete metric differs from this asymptotic form by exponentially small terms.

It can be easily seen that the Heisenberg manifold (11) is a special case of this setup for $n = 2$. As we have already described in Section 4.6, Kobayashi points out that for $n = 2$ these gravitational instantons arise as certain degenerations of the $K3$ surface. The metric spheres at large t represent a collapse of a Nilmanifold to a flat T^2 , and the volume of a metric ball grows as $t^{4/3}$. The Ricci-flat metric on D is flat only in this case.

Consider the following ansatz compatible with the above remarks:

$$ds^2 = dt^2 + a^2(t)g_{ab} dx^a dx^b + c^2(t)(d\tau - 2A)^2. \tag{54}$$

Here g_{ab} is the complete Ricci-flat Kähler metric on D , $a, b = 1, \dots, 2(n - 1)$; τ is the periodic coordinate on the canonical bundle over D and A is a one-form that depends only on x^a such that its exterior derivative is proportional to the Kähler form on D , $dA = -\sigma J$, σ is constant.

The Einstein equations for (54) reduce to a system of second order ODEs for the functions $a(t)$ and $c(t)$:

$$0 = \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{c}}{ac} + (2n - 3) \left(\frac{\dot{a}}{a}\right)^2 + 2\sigma^2 \frac{c^2}{a^4}, \tag{55}$$

$$0 = \frac{\ddot{c}}{c} + 2(n - 1) \frac{\ddot{a}}{a}, \tag{56}$$

$$0 = \frac{\ddot{c}}{c} + 2(n - 1) \frac{\dot{a}\dot{c}}{ac} - 2(n - 1)\sigma^2 \frac{c^2}{a^4}. \tag{57}$$

We therefore look for solutions with polynomial dependence on the radial coordinate t of the form

$$c(t) = \mu t^{\lambda_1}, \quad a(t) = \nu t^{\lambda_2}, \tag{58}$$

where $\mu, \nu, \lambda_1, \lambda_2$ are constants. Substituting (58) into Eqs. (55)–(57) we find

$$0 = \lambda_2(\lambda_2 - 1) + \lambda_1\lambda_2 + (2n - 3)\lambda_2^2 + 2\sigma^2 \frac{\mu^2}{\nu^4} t^{2(1+\lambda_1-2\lambda_2)}, \tag{59}$$

$$0 = \lambda_1(\lambda_1 - 1) + 2(n - 1)\lambda_2(\lambda_2 - 1), \tag{60}$$

$$0 = \lambda_1(\lambda_1 - 1) + 2(n - 1)\lambda_1\lambda_2 - 2(n - 1)\sigma^2 \frac{\mu^2}{\nu^4} t^{2(1+\lambda_1-2\lambda_2)}. \tag{61}$$

In accordance with ansatz (58) the Eqs. (59)–(61) must reduce to algebraic equations for the constants $\mu, \nu, \lambda_1, \lambda_2$. Hence we find that λ_1 and λ_2 must satisfy

$$\lambda_1 = 2\lambda_2 - 1. \tag{62}$$

Substituting (62) into (60) we obtain a quadratic equation for λ_2 :

$$(n + 1)\lambda_2^2 - (n + 2)\lambda_2 + 1 = 0, \tag{63}$$

which is solved by

$$\lambda_2^{(1)} = \frac{1}{n+1}, \quad \lambda_2^{(2)} = 1. \tag{64}$$

From (62) we have

$$\lambda_1^{(1)} = -\frac{n-1}{n+1}, \quad \lambda_1^{(2)} = 1. \tag{65}$$

We discard the pair $(\lambda_1^{(2)}, \lambda_2^{(2)}) = (1, 1)$ since it satisfies both (59) and (61) only for $n = 0$. We are thus left with the other pair of solutions $(\lambda_1, \lambda_2) \equiv (\lambda_1^{(1)}, \lambda_2^{(1)}) = (-(n-1)/(n+1), 1/(n+1))$. Let us now find the constants μ, ν and σ . From (61), or equivalently from (59), we have

$$\sigma^2 \frac{\mu^2}{\nu^4} = \frac{1}{(n+1)^2}. \tag{66}$$

The values of μ and ν for $n = 2$ may be read off from the Heisenberg metric (11): $\mu = (3/2)^{-1/3}$ and $\nu = (3/2)^{1/3}$. With these values (66) gives

$$\sigma = \frac{1}{2}. \tag{67}$$

Since parametrization of the one-form A should not depend on the dimension of M , we are compelled to choose constants μ and ν to satisfy (66) with $\sigma = 1/2$ and consistent with their values for $n = 2$. An appropriate choice is

$$\mu = \left(\frac{n+1}{2n}\right)^{-(n-1)/(n+1)}, \quad \nu = \left(\frac{n+1}{2n}\right)^{1/(n+1)}. \tag{68}$$

Absorbing the constant $-\sigma = -1/2$ into the definition of the one-form A we may now write down the asymptotic metric for the $2n$ -dimensional BKTY gravitational instanton:

$$\begin{aligned} ds^2 = dt^2 + \left(\frac{n+1}{2n}t\right)^{-2(n-1)/(n+1)} (d\tau + A)^2 \\ + \left(\frac{n+1}{2n}t\right)^{2/(n+1)} g_{ab} dx^a dx^b, \end{aligned} \tag{69}$$

where now dA is precisely the Kähler form on D .

Metric (69) indeed has the volume growth predicted by Kobayashi:

$$\text{Vol} \sim \int t^{-(n-1)/(n+1)} \cdot t^{2(n-1)/(n+1)} dt \sim t^{2n/(n+1)}.$$

7.2. Bianchi type I

Metrics of Kasner type (7) exist in arbitrary number of dimensions; the metric becomes

$$ds^2 = dt^2 + t^{2\alpha_1}(dx^1)^2 + t^{2\alpha_2}(dx^2)^2 + \dots + t^{2\alpha_n}(dx^n)^2,$$

where

$$\sum_{k=1}^n \alpha_k = 1 = \sum_{k=1}^n \alpha_k^2. \tag{70}$$

Just as the Kasner metric (7), these metrics are not BPS.

7.3. Bianchi type II

A particular case of the asymptotic BKTY $2n$ -dimensional metric (69) is that whose isometry group is the higher-dimensional Bianchi type II group. In this case the arbitrary Calabi–Yau $(2n - 2)$ -dimensional metric $g_{ab} dx^a dx^b$ is the flat metric

$$g_{ab} dx^a dx^b = \sum_{i=1}^{n-1} (\sigma_i^1)^2 + (\sigma_i^2)^2,$$

and the term involving the connection on the canonical bundle over $g_{ab} dx^a dx^b$ is simply $(\sigma^3)^2$; where $\{\sigma_i^1, \sigma_i^2, \sigma^3\}$ are the one-forms defined in (52).

To find the Kähler potential generating the metric (69) for the special case of Bianchi type II isometry group we solve the Monge–Ampère equation (34) with $\Lambda = 0$:

$$(K')^{n-1} K'' = (-1)^{n-1}. \tag{71}$$

It is sufficient to know $K'(f)$ where $f = z + \bar{z} - \sum_i |w^i|^2$ since the higher-dimensional Bianchi type II metric is expressed in terms of K' and K'' as follows:

$$ds^2 = -K' \sum_i dw^i \wedge d\bar{w}^i + K'' \left(dz - \sum_i \bar{w}^i dw^i \right) (d\bar{z} - w^i d\bar{w}^i),$$

which after coordinate redefinitions (35) with $\{w, x, y\}$ replaced by $\{w^i, x^i, y^i\}$ becomes

$$ds^2 = -K' \sum_i ((dx^i)^2 + (dy^i)^2) + K'' df^2 + K'' \left(d\tau - \sum_i x^i dy^i \right)^2. \tag{72}$$

Integrating Eq. (71) we find

$$K' = (-1)^{(n-1)/n} (nf)^{1/n}$$

and hence

$$K'' = (-1)^{(n-1)/n} (nf)^{-(n-1)/n}.$$

To compare with the metric we have obtained by solving the self-duality equations let us define a new radial coordinate t such that

$$f = \left(\frac{n+1}{2n} t \right)^{2n/(n+1)}.$$

Written in terms of t the metric (72) is the same as (69) up to an overall constant factor.

7.4. Bianchi type VII₀ and VI₀

The analysis based on solving the Monge–Ampère equation (45) and (50) may be extended to arbitrary number of dimensions. In fact in the case of Bianchi type VII₀ it was done by Calabi [36]. If M has real dimension $2n$ with complex coordinates z^a , $a = 1, \dots, n$ and one assumes, as was done in Section 6.1, that the Kähler potential K is independent of the imaginary parts of $z^a = u^a + iv^a$ and is solely a function of $r \equiv \sqrt{(u^1)^2 + \dots + (u^n)^2}$, the resulting metric will have the isometry group $E(n)$. The Monge–Ampère equation reduces to

$$\left(\frac{K'}{r}\right)^{n-1} K'' = 1.$$

It is solved by

$$K(r) = \int_0^r (c + r^{1/n}) dr, \quad c = \text{const.}$$

The manifold is a higher-dimensional vacuum Bianchi type VII₀ metric whose isometry group is $E(n)$. It is incomplete. Arguments analogous to the ones just given extend to the Bianchi type VI₀ metrics.

8. Other Bianchi types: Bianchi type III

We have not attempted here to survey all known cohomogeneity one Einstein metrics. Even in four dimensions this would be a formidable task. Some pertinent references in that case are [5,12]. However we would like to comment on the Bianchi type III situation since it may well prove relevant for various applications of Anti-de Sitter space–time.

The most general diagonal Lorentzian Bianchi type III local solution is given in [38]. A simple analytic continuation of the metric in [38] gives a Riemannian metric with negative scalar curvature $\Lambda < 0$:

$$ds^2 = \frac{3}{\Lambda} \left(\frac{d\tau^2}{\sinh^2 \tau} + \frac{d\Omega_{-1}^2}{\sinh^2 \tau} + \frac{d\alpha^2}{\tanh^2 \tau} \right), \tag{73}$$

which is presumably the most general local solution with this signature. Setting

$$t = \log \tanh \frac{\tau}{2}$$

gives

$$ds^2 = \frac{3}{\Lambda} (dt^2 + \sinh^2 t d\Omega_{-1}^2 + \cosh^2 t d\alpha^2).$$

In (73) $d\Omega_{-1}^2$ is the standard metric on \mathcal{H}^2 . The isometry group of the manifold is therefore $SO(2, 1) \times SO(2)$. The group $SO(2, 1)$ has a two-dimensional subgroup \tilde{G}_2 which acts transitively on \mathcal{H}^2 and combined with $SO(2)$ we get a three-dimensional Lie group with

three-dimensional orbits whose Lie algebra corresponds to Bianchi type III. Explicitly we consider \mathcal{H}^2 in horospheric coordinates (x, y) :

$$\frac{dx^2 + dy^2}{y^2}$$

and $\tilde{G}_2 = \mathbb{R} \ltimes \mathbb{R}$ is generated by $\partial/\partial x$ and $x\partial/\partial x + y\partial/\partial y$.

In fact the metric (73) is that of hyperbolic four-space \mathcal{H}^4 (cf. Section 5.3). This may be seen by isometrically embedding (73) into $\mathbb{F}^{4,1}$ as

$$(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = 1,$$

where

$$\begin{aligned} X^3 &= \cosh t \cos \alpha, & X^4 &= \cosh t \sin \alpha, & X^0 + X^1 &= \frac{1}{y} \sinh t, \\ X^0 - X^1 &= \left(y + \frac{x^2}{y} \right) \sinh t, \\ X^2 &= \frac{x}{y} \sinh t. \end{aligned}$$

It is now obvious that the Bianchi type III solution (73) may be extended to $(n+2)$ dimensions by replacing the metric on \mathcal{H}^2 by that on \mathcal{H}^n . The group \tilde{G}_2 is replaced by the group $\tilde{G}_n = \mathbb{R} \ltimes \mathbb{R}^{n-1}$ generated by $\partial/\partial x^i$ and $x^i \partial/\partial x^i + y \partial/\partial y$, $i = 1, \dots, n-1$, then the generalization of the Bianchi type III group is $(\mathbb{R} \ltimes \mathbb{R}^{n-1}) \times SO(2)$.

9. Conclusions

In this paper we have studied various solutions of M-theory having the symmetries of a domain wall. Our most important example (10) is BPS and is based on the Bianchi type II group, otherwise known as Nil or Heisenberg. Usually this is regarded as an orbifold solution with singularity. We have shown how the singularity may be resolved to give a complete non-singular solution representing a single-sided domain wall. We have also shown how this example may be generalized to higher dimensions.

Finally, we have considered a number of related solutions, some BPS, both in four and higher dimensions which we believe may be relevant to, for example, the AdS/CFT correspondence and other future applications of 11-dimensional supergravity.

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